PRACTICE FINAL (CHRIST) - BRIEF SOLUTIONS

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(1a) $f'(2) = 1 + e^2$, so $\boxed{y - (2 + e^2) = (1 + e^2)(x - 2)}$

(1b) $\boxed{\frac{1}{2\sqrt{3+\ln(\ln(x))}} \frac{1}{\ln(x)} \frac{1}{x}}$

(1c) $\boxed{-1}$ (use l'Hopital's, or notice that it's just the reciprocal of $cos'(\frac{\pi}{2})$)

(1d) 1) $y = x^{\cos(x)}$
2) $\ln(y) = \cos(x)\ln(x)$
3) $\frac{y'}{y} = -\sin(x)\ln(x) + \frac{\cos(x)}{x}$
4) $y' = x^{\cos(x)} \left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$

(1e) $\boxed{\sqrt{|\sin(x) + \cos(x)|} + C}$

(1f) $\cos(2x)(2)\sin^{-1}(\sin(2x)) = \cos(2x)(2)(2x) = \boxed{4x\cos(2x)}$

(1g) $\boxed{\frac{\sin(x)^2}{2} + C}$ (if you use the substitution $u = \sin(x)$) or $\boxed{-\frac{\cos(2x)}{4} + C}$ (if you notice that $\sin(x)\cos(x) = \frac{\sin(2x)}{2}$

(2a) $\sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 \boxed{\frac{\pi}{2}}$

(2b) $-1 + 4 - 9 + 16 = \boxed{10}$

(2c) $\boxed{0}$ (odd function!)

(2d) Don't worry about this one, won't appear on the exam, but you'll see this in Math 1B! Let $x = \sin(\theta)$, so $dx = \cos(\theta)d\theta$, and so the integral becomes:

$$\begin{aligned}
\int (1 - \sin^2(\theta))^{-\frac{3}{2}}(\cos(\theta))d\theta &= \int (\cos^2(\theta))^{-\frac{3}{2}}(\cos(\theta))d\theta \\
&= \int \cos(\theta)^{-3}\cos(\theta)d\theta \\
&= \int \cos(\theta)^{-2}d\theta \\
&= \int \sec^2(\theta)d\theta \\
&= \tan(\theta) + C \\
&= \tan(\sin^{-1}(\theta)) + C
\end{aligned}$$

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(2e) This is **very** hard!

$$\begin{aligned}
\lim_{x\to\infty}(x+\sqrt[3]{x})^{\frac{2}{3}}-x^{\frac{2}{3}} &= \lim_{x\to\infty} x^{\frac{2}{3}}\left(\frac{(x+\sqrt[3]{x})^{\frac{2}{3}}}{x^{\frac{2}{3}}}-1\right)\\
&= \lim_{x\to\infty} x^{\frac{2}{3}}\left(\left(\frac{x+\sqrt[3]{x}}{x}\right)^{\frac{2}{3}}-1\right)\\
&= \lim_{x\to\infty} x^{\frac{2}{3}}\left(\left(1+x^{-\frac{2}{3}}\right)^{-\frac{2}{3}}-1\right)\\
&= \lim_{x\to\infty} \frac{\left(1+x^{-\frac{2}{3}}\right)^{-\frac{2}{3}}-1}{x^{-\frac{2}{3}}}\\
&\overset{H}{=} \lim_{x\to\infty} \frac{-\frac{2}{3}(1+x^{-\frac{2}{3}})^{-\frac{5}{3}}(-\frac{2}{3}x^{-\frac{5}{3}})}{-\frac{2}{3}x^{-\frac{5}{3}}}\\
&= \lim_{x\to\infty} -\frac{2}{3}(1+x^{-\frac{2}{3}})^{-\frac{5}{3}}\\
&= -\frac{2}{3}
\end{aligned}$$

(2f) $\boxed{e^{\frac{4}{9}}+e^{\frac{16}{9}}+e^{2}}$

(2g)

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2\left(1+\frac{i^2}{n^2}\right)}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{1}{\left(1+\left(\frac{i}{n}\right)^2\right)}\\
&= \int_0^1 \frac{1}{1+x^2}dx\\
&= \tan^{-1}(1)-\tan^{-1}(0)\\
&= \frac{\pi}{4}-0\\
&= \frac{\pi}{4}
\end{aligned}$$

(2h) Ignore this, won't be on the exam!

(3) 1) $V=\frac{\pi}{3}r^2h$, $V(h)=\frac{\pi}{3}(1-h^2)(h)dh=\frac{\pi}{3}h-\frac{\pi}{3}h^3dh$

2) Constraint $0\leq h\leq 1$ (because of the requirement that $r^2+h^2=1$)

3) $V'(h)=\frac{\pi}{3}-\pi h^2=0 \iff h=\sqrt{\frac{1}{3}}$

4) $V(0)=0$, $V(1)=0$, and $V(\sqrt{\frac{1}{3}})=\frac{2\pi}{9\sqrt{3}}$, so by the closed interval method, $\boxed{V(\sqrt{\frac{1}{3}})=\frac{2\pi}{9\sqrt{3}}}$ is the maximum volume!

(4) $f(x) = 4 + xe^{-\frac{1}{2x}}$
   1) Domain: $x \neq 0$
   2) No $y-$intercepts, $x-$ intercept exists by the IVT
   3) No symmetry
   4) No horizontal asymptotes, Vertical Asymptote $x = 0$:

$$\lim_{x\to 0^-} f(x) = 4 + \lim_{x\to 0^-} \frac{e^{-\frac{1}{2x}}}{\frac{1}{x}} \stackrel{H}{=} 4 + \lim_{x\to 0^-} \frac{\frac{1}{2x^2}e^{-\frac{1}{2x}}}{-\frac{1}{x^2}} = 4 + \lim_{x\to 0^-} -\frac{1}{2}e^{-\frac{1}{2x}} = 4 - \infty = -\infty$$

   5) f increasing on $(-\infty, -\frac{1}{2})$, decreasing on $(-\frac{1}{2}, 0)$ and increasing on $(0, \infty)$. (It might help to notice that $1 + \frac{1}{2x} = \frac{2x+1}{2x}$ and use a sign table) Local minimum $\boxed{4 - \frac{1}{2}e \text{ at } x = -\frac{1}{2}}$
   6) Concave down on $(-\infty, 0)$, Concave up on $(0, \infty)$
   7) Check with a calculator to see if your picture is correct!

(5) I am skipping this one as well, because it won't be on the exam!

(6) (a) Using the shell method (draw a good picture),

$$V = \int_1^3 2\pi\,|x - 0|\,(BIGGER - SMALLER)dx$$

Now, $|x - 0| = |x| = x$, and $(x-2)^2 + y^2 = 1 \iff y = \pm\sqrt{1 - (x-2)^2}$,
So BIGGER = $\sqrt{1-(x-2)^2}$ and SMALLER = $-\sqrt{1-(x-2)^2}$,
So BIGGER - SMALLER = $2\sqrt{1-(x-2)^2}$, and the volume becomes:

$$V = \int_1^3 2\pi x(2\sqrt{1-(x-2)^2})dx = \int_1^3 4\pi x\sqrt{1-(x-2)^2}dx$$

(b) First of all, let $u = x - 2$, so $x = u + 2$ then $du = dx$, and V becomes:

$$V = \int_{-1}^1 4\pi(u+2)\sqrt{1-u^2}du = 4\pi\int_{-1}^1 u\sqrt{1-u^2}du + 8\pi\int_{-1}^1 \sqrt{1-u^2}du = 0 + 8\pi\frac{\pi}{2} = 4\pi^2$$

Where the last integral is the sum of the integral of an odd function, and the area of a semicircle of radius 1.

(7) Don't worry about those, they won't be on the exam (but you might have true/false questions on the exam)